The 0–1 Knapsack Problem: A Continuous Generalized Convex Multiplicative Programming Approach

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ABSTRACT

In this work we propose a continuous approach for solving one of the most studied problems in combinatorial optimization, known as the 0–1 knapsack problem. In the continuous space, the problem is reformulated as a convex generalized multiplicative problem, a special class of nonconvex problems which involves the minimization of a finite sum of products of convex functions over a nonempty convex set. The product of any two convex positive functions is not necessarily convex or quasiconvex, and, therefore, the continuous problem may have local optimal solutions that are not global optimal solutions. In the *outcome space*, this problem can be solved efficiently by an algorithm which combines a relaxation technique with the procedure branch–and–bound. Some computational experiences are reported.

KEYWORDS. Knapsack Problem, Combinatorial Optimization, Global Optimization, Multiplicative Programming, Convex Analysis.

Main Area: Combinatorial Optimization

RESUMO

Neste trabalho propomos uma abordagem contínua para resolver um dos problemas mais estudados de otimização combinatória, conhecido como problema de mochila 0–1. No espaço contínuo, o problema é reformulado como um problema multiplicativo generalizado convexo, uma classe especial de problemas não–convexos que envolve a minimização de uma soma finita de produto de funções convexas sobre um conjunto convexo, compacto e não vazio. O produto de quaisquer duas funções convexas positivas não é necessariamente convexa ou quase–convexa, e, portanto, o problema contínuo pode ter soluções ótimas locais que não são soluções ótimas globais. No espaço dos objetivos, este problema pode ser eficientemente resolvido por um algoritmo que combina relaxação com uma técnica de *branch–and–bound*. Algumas experiências computacionais são relatadas.

PALAVRAS CHAVE. Problema de Mochila, Otimização Combinatória, Otimização Global, Programação Multiplicativa, Análise Convexa.

Área Principal: Otimização Combinatória

1. Introduction

A famous class of combinatorial optimization problems is known as the knapsack problem. In particular, this class of combinatorial optimization problems, characterizes a class of integer linear programming and are classified as NP–hard problems due to their complexity degree (Zhang and Geng, 1986 and Kellerere et al., 2004). Consider the situation where a mountaineer who is packing his knapsack for a mountain tour and has to decide which items he should take with him, among several items available and considering the limited knapsack capacity. The items have different weights and each of them would give the mountaineer a certain amount of comfort/benefit which is measured by positive value, known as utility. For obvious reasons, the goal is to maximize the total utility of the items taken without exceeding the prescribed knapsack capacity.

Given p items $x_1,\ x_2,\dots,\ x_p$, each x_i with weight (cost) $w_i \in \mathbb{R}$ and utility (value) $u_i \in \mathbb{R}$, and a knapsack capacity (budget) $C \in \mathbb{R}$. Then, the problem of maximizing the total utility of the items taken without exceeding the prescribed limit C can be formulated as

$$\left|\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{p} u_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{p} w_i x_i \leq C. \end{array}\right. \tag{1.1}$$

In the unbounded knapsack problem, there is no upper bound on the number of copies of each item which the mountaineer can take with him, but, naturally, there is a limited supply of each item. In other words, let the integer value c_i be the upper bound on the number of copies of item x_i $(i = 1, 2, \dots, n)$. In this case, the knapsack problem is bounded and can be formulated as

$$\left|\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{p} u_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{p} w_i x_i \leq C \\ & x_i \in \{0, 1, 2 \dots, c_i\}, \quad i = 1, 2, \dots, p. \end{array}\right. \tag{1.2}$$

In particular, when each item is unique $(c_i = 1,\ i = 1, 2, \dots, p)$, the bounded knapsack problem may be reformulated as follows and is known as the 0–1 knapsack problem

$$\left|\begin{array}{ll} \text{maximize} & \displaystyle\sum_{i=1}^{p} u_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{p} w_i x_i \leq C \\ & x_i \in \{0, 1\}, \quad i = 1, 2, \dots, p. \end{array}\right. \tag{1.3}$$

The class of knapsack problems is very wide and includes many subproblems, and, there is no text in the literature of combinatorial optimization that covers and fully treats it. However, there is a great number of books and research texts that cover several classic problems of the family, treat some more specific generalizations of the problems, or give a profound introduction (see e.g., Ibraki 1987, Pisinger 1995, Dudzinski and Walukiewicz 1987, Martello and Toth 1990 and Syslo et al. 1983).

Several important problems arising in Operations Research, Mathematical Programming, Engineering, Economics, Packing and Planning are modeled in a convenient way by the knapsack problems of the form (1.2)–(1.3) (Pisinger, 1995). For example, suppose that one must allocate a single scarce recourse among multiple contenders for this resource while obtaining some sort of profit from an optimal configuration. Another interesting applications appear in, for example, the diet problem (Sinha and Zoltners, 1979), Bin–packing problem (Sinuary–Stern and Winer, 1994), cargo loading, project selection, cutting stock, budget control, financial management (see Salkin and Kluyver, 1975 for a detailed discussion).

The literature in the knapsack problems has been dominated by the the analysis of problem with binary variables, the 0–1 knapsack problem, since the pioneering work of Dantzing in the late 50's (Dantzing, 1957). Since then, a number of different approaches for solving the knapsack problems have been proposed. The 0–1 knapsack problem has attracted special interest.

As knapsack problems (in particular 0–1 knapsack problem) are classified as NP–hard problems due to their complexity degree, there is no exact solution methods other than the enumeration space approaches. However, a wide variety of inexact approaches, including branch–and–bound, dynamic programming, state space relaxation and preprocessing, have been proposed in the literature of integer programming for solving knapsack problems (see Ibraki, 1987).

The principal objective of this paper is to introduce a continuous optimization technique for solving the 0–1 knapsack problem, one of the classic problems of combinatorial optimization. The 0–1 knapsack problem, is perhaps, the most important knapsack problem and one of the most studied problems of discrete optimization, once it can be seen as the most simple problem of integer linear programming, it appears as a sub–problem in many other complex problems and it represents a very wide range of practical situations. In the continuous space, the problem is reformulated as a convex generalized multiplicative problem, a special class of nonconvex problems which involves the minimization of a finite sum of products of concave functions over a nonempty convex set. In the *outcome space*, a branch–and–bound algorithm is proposed for solving such problem.

The paper is organized in six sections, as follows. In Section 2, the 0–1 knapsack problem is reformulated in the continuous space as an equivalent convex generalized multiplicative programming problem. In Section 3, the equivalent problem is reformulated in the outcome space, and an outer approximation approach is outlined. In Sections 4, the relaxation branch–and–bound algorithm is derived. Some computational experiences with the method described in Section 4 are reported in Section 5. Conclusions are presented in Section 6.

Notation. Throughout this paper, the set of all n-dimensional real vectors is represented as $\mathbb{R}^n$. The sets of all nonnegative and positive real vectors are denoted as $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively. Inequalities are meant to be componentwise: given $x, y \in \mathbb{R}^n_+$, then $x \geq y$ ($x - y \in \mathbb{R}^n$) implies $x_i \geq y_i, i = 1, 2, ..., n$. Accordingly, $x > y$ ($x - y \in \mathbb{R}^n_{++}$) implies $x_i > y_i, i = 1, 2, ..., n$. The standard inner product in $\mathbb{R}^n$ is denoted as $\langle x, y \rangle$. If $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$ is defined on Ω, then $f(\Omega) := \{f(x) \ : \ x \in \Omega\}$. The symbol := means *equal by definition*.

2. The Continuous Reformulation

A nontraditional approach for solving discrete programming problems (in particular, the 0–1 knapsack problem) can be that of transforming the problem into an equivalent continuous problem. These solution methods are based on different characterizations or reformulations of the considered problems in a continuous space and involve analytic and algebraic techniques (see Pardalos and Rosen 1987, Leyffer 1993, Horst and Tuy 1996, Pardalos 1996, Du and Pardalos 1997, Pardalos 1998, Horst et al. 2000, Pardalos et al. 2006, Mangasarian 2009, Murray and Ng 2010). In particular, Pardalos (1996) and Pardalos et al. (2006) give a brief overview of some continuous approaches to some discrete optimization problems.

In this section, a continuous reformulation for solving a given 0–1 knapsack problem is considered; we show that this continuous reformulation has the same global maximizer of the original 0–1 knapsack problem. Hence, the optimal solution of the original 0–1 knapsack problem can be obtained by solving this specific continuous problem. Consider the 0–1 knapsack problem (KP), formulated as follow:

$$\text{KP}\quad\left|\begin{array}{ll}\text{maximize} & \displaystyle\sum_{i=1}^{p} u_i x_i \\ \text{subject to} & \displaystyle\sum_{i=1}^{p} w_i x_i \leq C \\ & x_i \in \{0,1\}, \quad i = 1,2,\ldots,p.\end{array}\right.$$

As noted knapsack problems are generally NP–hard and yet there are few successful solution methods. Dropping the binary variables constraints, $x_i \in \{0,1\}, \quad i = 1,2,\ldots,p$, will make the computational effort to increase slowly with the size of the problem. Perhaps, one of the most important properties of the 0–1 knapsack problem (the knapsack problems, in general) is that the relaxed problem (continuous version of the problem), where the binary variables constraints are relaxed to $x_i \in [0,1], \quad i = 1,2,\ldots,p$, is so fast to solve.

In Dantzig (1957), a solution method is proposed for solving the continuous 0–1 knapsack problem. The proposed technique by Dantzig is based on the ordering the items according to their profit–to–weight ratio and using a greedy algorithm for filling the knapsack. It can be seen that, having solved the continuous 0–1 knapsack problem, a few decision variables may be changed in order to obtain the optimal integer solution (Pisinger, 1995). Moreover, the original 0–1 knapsack problem has a very large number of local minimizers, in general, that it makes the continuous problem less complicated to work on, if there is an efficient solution method available to solve it (Murray and Ng, 2010). For these reasons, and, among others, it would seen attractive if the 0–1 knapsack problem would be replaced by an equivalent relaxed problem in continuous variables.

Based on these observations, initially, the original 0–1 knapsack problem must be relaxed. Perhaps, the most simple way to do it is to add constraints $x_i \in [0,1], \quad i = 1,2,\ldots,n$ and $x_i(x_i-1) = 0, \quad i = 1,2,\ldots,p$. Clearly, $x_i = 0$ or $x_i = 1$ for $i = 1,2,\ldots,p$. The problem is that a nonconvex optimization problems may have local optimal solutions that are not global optimal solutions. Another way to do it is to add the penalty term $\sum_i x_i(1-x_i)$ to the objective function with an arbitrarily large positive penalty parameter $M > 0$. The continuous 0–1 knapsack problem (CKP) then becomes

$$\text{CKP}\quad\left|\begin{array}{ll}\text{maximize} & \displaystyle\sum_{i=1}^{p} u_i x_i - M\sum_{i=1}^{p} x_i(1-x_i) \\ \text{subject to} & \displaystyle\sum_{i=1}^{p} w_i x_i \leq C \\ & x_i \in [0,1], \quad i = 1,2,\ldots,np.\end{array}\right.$$

Problem (CKP) is a maximization problem, therefore, in the optimality the penalty term $\sum_i x_i(1-x_i)$ must be zero since the penalty parameter M is an arbitrarily large positive number. This observation together with the constraint $x_i \in [0,1], \quad i = 1,2,\ldots,p$ imply that $x_i \in \{0,1\}, \quad i = 1,2,\ldots,p$. Hence, we have the following equivalence theorem.

Theorem 2.1 (Equivalence Theorem) (KP) *is equivalent to* (CKP) *where M is an arbitrarily large positive number.*

Since each $x_i(1-x_i)$ is concave in $x_i, \quad i = 1,2,\ldots,p$, the objective function in (CKP) is convex in $x_1, x_2, \ldots, x_p$. It is known that the maximum of a convex function over a compact convex polyhedral set is attained at one of its finitely many extreme points (Horst et al. 2000). It also can be seen that the set of the feasible solutions of (CKP) in standard form is convex and every integer feasible solution is an extreme point of this set.

It would seen that the vertices enumeration approaches can be used and applied but the computational complexity of problems relating to the enumeration of all the vertices of a convex polyhedral set defined by linear inequalities is superpolynomial (Dyer, 1983), which it leads to other interesting approaches that involve solution methods to generalized multiplicative programming problems, first introduced in Konno at al. (1994). Instead of the convex maximization problem (CKP), we consider the following equivalent concave minimization problem

$$\text{ECKP}\left|\begin{array}{ll}\text{minimize} & \big(-\sum_{i=1}^{p} u_i x_i + \alpha\big) + M\sum_{i=1}^{p} x_i(1-x_i)\\ \text{subject to} & \sum_{i=1}^{p} w_i x_i \leq C\\ & x_i \in [0,1], \quad i = 1, 2, \ldots, p.\end{array}\right.$$

where $\alpha \in \mathbb{R}_{++}$ is such that $(-\sum_{i=1}^{p} u_i x_i + \alpha) > 0$. It is well known that problem (ECKP) has the same solutions as (CKP), but it is better conditioned since all functions involved are positive over its constraint set.

In the next section, we propose an outcome space approach for globally solving the equivalent convex generalized multiplicative programming problem (ECKP), which involves the minimization of a finite sum of products of convex functions over a nonempty compact convex polyhedral set. It is shown that this nonconvex minimization problem can be reformulated as an indefinite quadratic problem with infinitely many linear inequality constraints.

3. Outcome Space Formulation of (ECKP)

This section is concerned with the convex generalized multiplicative programming problem (ECKP), a special class of problems of minimizing an arbitrary finite sum of products of two convex functions over a compact convex set, a problem originally proposed in Konno et al. (1994). Consider the multiplicative programming problem

$$\min_{x\in\Omega}\, v(x) = \min_{x\in\Omega} f_1(x) + \sum_{i=1}^{p} f_{2i}(x) f_{2i+1}(x), \tag{3.1}$$

where $f_i : \mathbb{R}^n \to \mathbb{R},\ i = 1, 2, ..., m,\ m = 2p + 1$, are convex functions defined on $\mathbb{R}^n$. It is also assumed that Ω is a nonempty compact convex set and that $f_1, f_2, ..., f_m$ are positive functions over Ω.

The product of any two convex positive functions is not necessarily convex or quasi–convex, and, therefore, problem (3.1) may have local optimal solutions that are not global optimal solutions. In nonconvex global optimization, problem (3.1) has been referred as the generalized convex multiplicative problem. Important problems in engineering, financial optimization, microeconomics, geometric design and economics, among others, rely on mathematical optimization problems of the form (3.1). See (Ashtiani, 2012) for a detailed discussion about generalized multiplicative programming problems.

In the last decade, many efficient solution algorithms have been proposed for globally solving the (generalized) multiplicative programming problems class and its several particular cases. A number of multiplicative programming approaches for solving this problem in the outcome space have been proposed. In (Konno et al., 1994) the problem is projected in the outcome space, where the problem has only m variables, and then solved by an outer approximation algorithm. In (Oliveira and Ferreira, 2010) the problem is projected in the outcome space following the ideas introduced in (Oliveira and Ferreira, 2008), reformulated as an indefinite quadratic problem with infinitely many linear inequality constraints, and then solved by an efficient relaxation–constraint enumeration algorithm. In (Ashtiani and Ferreira, 2011) the authors address the closely related problem of *maximizing* the same objective function, but with $f_i,\ i = 1, 2, \ldots, m$ concave, rather than convex positive functions over Ω. More recently, a number of branch–and–bound techniques have also been proposed (in particular, Ashtiani (2012) gives an overview of some these approaches). In fact, generalized convex and generalized concave multiplicative problems are found in the fields of quadratic, bilinear and linear zero–one optimization.

The outcome space approach for solving problem (ECKP) is inspired in a similar approach recently introduced in (Oliveira and Ferreira, 2010) and (Ashtiani and Ferreira, 2011) for solving the classical convex generalized multiplicative problems (3.1). Let Ω the constraint set of (ECKP) and define $f_1 := (-\sum_{i=1}^{p} u_i x_i + \alpha)$, $f_{2i} := x_i$ and $f_{2i+1} := (1 - x_i)$ for $i = 1, 2, \ldots, p$. The objective function in (ECKP) can be written as the composition $u(f(x))$, where $u : \mathbb{R}^m \to \mathbb{R}$, $m = 2p + 1$, is defined by

$$u(y) := y_1 + M \sum_{i=1}^{p} y_{2i} y_{2i+1}.$$

The function u can be viewed as a particular aggregating function for the problem of minimizing the vector-valued objective $f := (f_1, Mf_2, f_3, \ldots, Mf_{2p}, f_{2p+1})$ over Ω (Yu, 1985). The image of Ω under f, $\mathcal{Y} := f(\Omega)$, is the outcome space associated with problem (ECKP). Since f is positive over Ω, it follows that u is strictly increasing over $\mathcal{Y}$ and any optimal solution of (ECKP) is Pareto–optimal or efficient (Yu, 1985). It is known from the multiobjective programming literature that if $x \in \Omega$ is an efficient solution of (ECKP), then there exists $w \in \mathbb{R}^m_+$ (different from what was considered in Section 1–2) such that x is also an optimal solution of the convex programming problem

$$\min_{x \in \Omega} \langle w, f(x) \rangle. \tag{3.2}$$

Conversely, if $x(w)$ is any optimal solution of (3.2), then $x(w)$ is efficient for (ECKP) if $w \in \mathbb{R}^m_{++}$. By defining

$$\mathcal{W} := \Big\{ w \in \mathbb{R}^m_+ \ : \ \sum_{i=1}^{m} w_i = 1 \Big\},$$

the efficient set of (ECKP), denoted as $\mathrm{effi}(\Omega)$, can be completely generated by solving (3.2) over $\mathcal{W}$. The outcome space formulation of problem (ECKP) is simply

$$\min_{y \in \mathcal{Y}} \ u(y) := y_1 + M \sum_{i=1}^{p} y_{2i} y_{2i+1}. \tag{3.3}$$

The solution approaches which aim at solving problem (ECKP) by solving its equivalent problem (3.3) in the outcome space basically differ in the way of representing the (generally) non-convex set $\mathcal{Y}$. In (Oliveira and Ferreira, 2010) a suitable representation is derived with basis on the following convex analysis result. See (Lasdon, 1970) for a proof.

Lemma 3.1 *Given $y \in \mathbb{R}^m$, the inequality $f(x) \le y$ has a solution $x \in \Omega$ if and only if y satisfies*

$$\min_{x \in \Omega} \langle w, f(x) \rangle \le \langle w, y \rangle \quad \textit{for all} \ \ w \in \mathcal{W}.$$

or, equivalently,

$$\max_{x \in \Omega} \langle w, f(x) - y \rangle \ge 0 \quad \textit{for all} \ \ w \in \mathcal{W}. \tag{3.4}$$

The main theoretical result of this paper consists in showing that problem (3.3) admits an equivalent formulation with a convex feasible region.

Theorem 3.2 *Let $y^\star$ be an optimal solution of problem*

$$\min_{y \in \mathcal{F}} \; u(y) \;:= y_1 + M \sum_{i=1}^{p} y_{2i} y_{2i+1} \tag{3.5}$$

where $\mathcal{F} := \mathcal{Y} + \mathbb{R}^m_+$. Then $y^\star$ is also an optimal solution of (3.3). Conversely, if $y^\star$ solves (3.3), then $y^\star$ also solves (3.5).

Proof. Since for any $x \in \Omega,\; y = f(x)$ is feasible for (3.5), the feasible set of (3.5) contains the feasible set of (3.3). Therefore, the optimal value of (3.5) is a upper bound for the optimal value of (3.3). If $y^\star$ solves (3.5), then

$$\min_{x \in \Omega} \langle w, f(x) - y \rangle \leq 0, \quad \text{for all } \; w \in \mathcal{W},$$

and by Lemma 3.1 there exists $x^\star \in \Omega$ such that $f(x^\star) \leq y^\star$. Actually, $f(x^\star) = y^\star$. Otherwise, the feasibility of $f(x^\star)$ for (3.5) and the positivity of u over $\mathcal{F}$ would contradict the optimality of $y^\star$. Since $f(x^\star)$ is feasible for (3.3), we conclude that $y^\star$ also solves (3.3). The converse statement is proved by using similar arguments. □

4. Relaxation Procedure

Problem (3.5) has a small number of variables, but infinitely many linear inequality constraints. An adequate approach for solving (3.5) is relaxation. The relaxation algorithm evolves by determining y^k, a global maximizer of u over an outer approximation $\mathcal{F}^k$ of $\mathcal{F}$ described by a subset of the inequality constraints (3.4), and then appending to $\mathcal{F}^k$ only the inequality constraint most violated by y^k. The most violated constraint is found by computing

$$\theta(y) \;:= \max_{w \in \mathcal{W}} \; \phi_y(w), \tag{4.1}$$

where

$$\phi_y(w) := \min_{x \in \Omega} \langle w, f(x) - y \rangle. \tag{4.2}$$

Maximin problems as the one described by (4.1) and (4.2) arise frequently in optimization, engineering design, optimal control, microeconomic and game theory, among other areas.

Lemma 4.1 *$y \in \mathbb{R}^m$ satisfies the inequality system (3.4) if and only if $\theta(y) \leq 0$.*

Proof. If $y \in \mathbb{R}^m$ satisfies the inequality system (3.4), then $\min_{x \in \Omega} \langle w, f(x) - y \rangle \leq 0$ for all $w \in \mathcal{W}$, implying that $\theta(y) \leq 0$. Conversely, if $y \in \mathbb{R}^m$ does not satisfy the inequality system (3.4), then $\min_{x \in \Omega} \langle w, f(x) - y \rangle > 0$ for some $w \in \mathcal{W}$, implying that $\theta(y) > 0$. □

Some useful properties of θ and ϕ are discussed in Oliveira and Ferreira (2008, 2010). The following geometric property of θ is proved in Oliveira and Ferreira (2010).

Theorem 4.2 *For any $y \in \mathbb{R}^m$, the value $\theta(y)$ is the optimal value of the following convex programming problem*

$$\left| \begin{array}{ll} \underset{x,\sigma}{\text{minimize}} & \sigma \\ \text{subject to} & f(x) \leq \sigma e + y \\ & x \in \Omega, \; \sigma \in \mathbb{R}. \end{array} \right. \tag{4.3}$$

where $\sigma \in \mathbb{R}$ and $e \in \mathbb{R}^m$ is the vector of ones.

If $x^\star$ and $w^\star$ are the primal and dual optimal solutions of problem (4.3), and $w^\star \in \mathbb{R}^m_{++}$, then the inequality constraint in (4.3) is active at $x^\star$. The case $\theta(y) > 0$ is more relevant for the analysis because the relaxation algorithm generates a sequence of infeasible points $y \notin \mathcal{F}$ converging to an optimal solution of (3.5) (see Lemma 4.1). In this case $\theta(y)$ is numerically equal to the infinity norm between y and $\mathcal{F}$. Some other useful properties of θ and ϕ are listed in Oliveira and Ferreira (2010).

Consider the initial polytope

$$\mathcal{F}^0 := \Big\{ y \in \mathbb{R}^m : 0 < \underline{y} \le y \le \overline{y} \Big\}, \tag{4.4}$$

where $\underline{y}$ and $\overline{y}$ are defined as $\underline{y}_i := \min_{x\in\Omega} f_i(x) > 0,\ \ \overline{y}_i := \max_{x\in\Omega} f_i(x),\ \ i = 1, 2, ..., m$. The computations of $\underline{y}$ and $\overline{y}$ demand m convex and m concave minimizations. While the computation of $\underline{y}$ is relatively inexpensive, the computation of $\overline{y}$ requires the solution of m nonconvex problems. However, the usual practice of setting the components of $\overline{y}$ sufficiently large has been successfully applied. It is readily seen that the minimization of u over $\mathcal{F}^0$ is achieved at $y^0 = \underline{y}$. The utopian point y^0 rarely satisfies the inequality system (3.4), that is, $\theta(y^0) > 0$, in general. By denoting as $w^0 \in \mathcal{W}$ the corresponding maximizer in (4.1), one concludes that y^0 is not in (most violates) the supporting negative half–space

$$\mathcal{H}^0_+ = \Big\{ y \in \mathbb{R}^m \ :\ \langle w^0, y\rangle \ge \langle w^0, f(x(w^0))\rangle \Big\}. \tag{4.5}$$

An improved outer approximation for $\mathcal{F}$ is $\mathcal{F}^1 = \mathcal{H}^0_+ \cap \mathcal{F}^0$. If y^1 that minimizes u over $\mathcal{F}^1$ is also such that $\theta(y^1) > 0$, then a new supporting positive half–space $\mathcal{H}^1_+$ is determined, the feasible region of (3.5) is better approximated by $\mathcal{F}^2 = \mathcal{F}^1 \cap \mathcal{H}^1_+$, and the process repeated. At an arbitrary iteration k of the algorithm, the following relaxed program is solved:

$$\min_{y\in\mathcal{F}^k} u(y). \tag{4.6}$$

4.1. A Relaxation Branch–and–Bound Algorithm

Problem (4.6) is actually a linearly constrained quadratic problem of the form

$$P_{\mathcal{F}^k} \quad \left|\ \begin{array}{ll} \text{minimize} & u(y) := y_1 + M\displaystyle\sum_{i=1}^{p} y_{2i}y_{2i+1} \\ \text{subject to} & A^{(k)}y \ge b^{(k)}, \\ & \underline{y} \le y \le \overline{y}, \end{array}\right.$$

where $A^{(k)} \in \mathbb{R}^{k\times m}, b^{(k)} \in \mathbb{R}^k, \underline{y} \in \mathbb{R}^m$ and $\overline{y} \in \mathbb{R}^m$ characterize the matrix representation of problem (4.6). The objective function in ($P_{\mathcal{F}^k}$) can be rewritten as

$$u(y) = \frac{1}{2}y^TQy + c^Ty, \tag{4.7}$$

where the characteristic equation of Q,

$$\det(\lambda I - Q) = \lambda\,\underbrace{(\lambda^2 - M^2)\cdots(\lambda^2 - M^2)}_{p \text{ times}} = 0,$$

has exactly p negative roots (eigenvalues) equal to $-M$, p positive roots equal to M, and one root equal to zero. This clearly implies the indefiniteness of Q, that is, ($P_{\mathcal{F}^k}$) is an indefinite quadratic programming problem. However, the characteristics of ($P_{\mathcal{F}^k}$) favour the application of the constraint enumeration method. Since Q has p positive eigenvalues, it follows that at least p constraints will be active at any local (global) solution of $(P_{\mathcal{F}^k})$, an optimal solution of $(P_{\mathcal{F}^k})$ occurs at the boundary of $\mathcal{F}^k$ and can be found by constraint enumeration (Horst et al. 2002). In this paper, global minimizers of $(P_{\mathcal{F}^k})$ are obtained as the limit of the optimal solutions of a sequence of special programs solved by using a rectangular branch–and–bound procedure.

Thus, the relaxation algorithm for globally solving the generalized multiplicative problem (ECKP) assumes the structure below.

Basic Algorithm

Step 0: Find $\mathcal{F}^0$ and set $k := 0$;

Step 1: Solve the generalized multiplicative problem $(P_{\mathcal{F}^k})$ using the **rectangular branch–and–bound** algorithm proposed as follow, obtaining y^k;

Step 2: Find $\theta(y^k)$ and w^k by solving problem (4.1)–(4.2). If $\theta(y^k) < \epsilon$, where $\epsilon > 0$ is a small tolerance, stop: y^k and $x(w^k)$ are ϵ–optimal solutions of (3.3) and (ECKP), respectively. Otherwise, define

$$\mathcal{F}^{k+1} := \{y \in \mathcal{F}^k \ : \ \langle w^k, y\rangle \geq \langle w^k, f(x(w^k))\rangle\},$$

set $k := k + 1$ and return to Step 1.

The infinite and finite convergence properties of Algorithm 1 are analogous to those exhibited by the algorithm derived in (Oliveira and Ferreira, 2010) for generalized multiplicative programming.

4.1.1. A Rectangular Branch–and–Bound Algorithm – Solving $(P_{\mathcal{F}^k})$

The relaxation algorithm evolves by determining y^k, a global minimizer $u(y)$ over an outer approximation $\mathcal{F}^k$ of $\mathcal{F}$, and then appending to $\mathcal{F}^k$ only the inequality constraint most violated by y^k.

Lower Bound

Let $\mathcal{R}$ denote either the initial rectangle $\mathcal{F}^0 := [\underline{y}, \overline{y}]$, or a subrectangle of it. In each subrectangle, any feasible point of $(P_{\mathcal{F}^k})$ provides an upper bound for the optimal value of $(P_{\mathcal{F}^k})$. In Adjiman et al. (1995), the authors discuss a convex lower bound for the bilinear term $y_{2i}y_{2i+1}$ inside a rectangular region $[y_{2i}^L, y_{2i}^U] \times [y_{2i+1}^L, y_{2i+1}^U]$, where $y_{2i}^L, y_{2i}^U, y_{2i+1}^L$ and y_{2i+1}^U are the lower and upper bounds on y_{2i} and y_{2i+1}, respectively. Bilinear terms of the form $y_{2i}y_{2i+1}$ are underestimated by introducing a new variable λ_i and two inequalities

$$y_{2i}y_{2i+1} \geq \max\Big\{y_{2i}^L y_{2i+1} + y_{2i+1}^L y_{2i} - y_{2i}^L y_{2i+1}^L, \\ y_{2i}^U y_{2i+1} + y_{2i+1}^U y_{2i} - y_{2i}^U y_{2i+1}^U\Big\}, \tag{4.8}$$

which depend on the bounds on y_{2i} and y_{2i+1}. Then, a lower bound for the optimal value of $(P_{\mathcal{F}^k})$ can be obtained by solving the following convex programming problem:

$$\begin{array}{|ll}
\underset{y,\lambda}{\text{minimize}} & \lambda_1 + M\displaystyle\sum_{i=1}^{p} \lambda_{i+1} \\
\text{subject to} & A^{(k)}y \geq b^{(k)} \\
& \lambda_1 \geq y_1, \\
& \lambda_{i+1} \geq y_{2i}^L y_{2i+1} + y_{2i+1}^L y_{2i} - y_{2i}^L y_{2i+1}^L, \quad i = 1, 2, \ldots, p, \\
& \lambda_{i+1} \geq y_{2i}^U y_{2i+1} + y_{2i+1}^U y_{2i} - y_{2i}^U y_{2i+1}^U, \quad i = 1, 2, \ldots, p, \\
& y \in \mathcal{R},
\end{array} \tag{4.9}$$

where $y_{2i}^L, y_{2i}^U, y_{2i+1}^L$ and y_{2i+1}^U $(i = 1, 2, \ldots, p)$ are the bounds on the variables y_{2i} and y_{2i+1} in some subrectangle $\mathcal{R}$. The rectangular branch–and–bound algorithm for globally solving the k-th outer approximation of the generalized multiplicative problem (ECKP) assumes the structure below.

Rectangular Branch–and–Bound Algorithm

Step 0: Find $\mathcal{F}^0$, let some accuracy tolerance $\epsilon BB > 0$ and the iteration counter $k = 0$.

Step 1: Define $\mathcal{L}_0 := \{\mathcal{F}^0\}$, and let L_0 and U_0 be a lower and an upper bound for the optimal value of problem $(P_{\mathcal{F}^k})$, which are found by solving problem (4.9) with $\mathcal{R} = \mathcal{F}^0$.

Step 2: While $U_k - L_k > \epsilon BB$,

i) Choose $\mathcal{R} \in \mathcal{L}_k$ such that the lower bound over $\mathcal{R}$ is equal to L_k;

ii) Split $\mathcal{R}$ along one of its longest edges into $\mathcal{R}_I$ and $\mathcal{R}_{II}$;

iii) Define

$$\mathcal{L}_{k+1} := (\mathcal{L}_k - \{\mathcal{R}\}) \cup \{\mathcal{R}_I, \mathcal{R}_{II}\},$$

iv) Compute lower and upper bounds for the optimal values of problems (4.9) with $\mathcal{R}_y = \mathcal{R}_y^I$ and (4.9) with $\mathcal{R}_y = \mathcal{R}_y^{II}$, set L_{k+1} and U_{k+1} as the minima lower and upper bounds over all subrectangles $\mathcal{R}_y \in \mathcal{L}_{k+1}$, and $k := k + 1$.

A similar convergence results for rectangular branch–and–bound algorithms can be found in (Benson, 2002).

5. Computational Experiments

The basic and the retangular branch–and–bound algorithms, which solve outer approximations of generalized multiplicative problems were coded in `MATLAB` (V. 7.0.1)/Optimization Toolbox (V. 4) and run on a personal Pentium IV system, 2.00 GHz, 2048MB RAM. The tolerances for the ϵ–convergences of algorithm was fixed at 10^{-3} while the tolerance for the convergence of the branch–and–bound algorithm was fixed at 0.05. In order to illustrate the convergence of the global optimization algorithms proposed, the following examples have been considered.

Example 5.1 *Let* $p = 5$ *(the number of items),* $w = [92\ 29\ 37\ 37\ 77]$ *(the weight vector) and* $u = [82\ 26\ 42\ 36\ 70]$ *(the utility vector). The solutions, in the term of the knapsack capacity,* C*, are reported in Table 1.*

Tabela 1: Example 5.1.

C	Optimal Solution	Optimal Value
100	(1,0,0,0,0)	82
155	(0,0,1,1,1)	148
183	(0,1,1,1,1)	174
225	(1,0,1,0,1)	194
270	(1,0,1,1,1)	230

Example 5.2 *Let* $p = 8$ *(the number of items),* $w = [15\ 28\ 32\ 21\ 25\ 18\ 20\ 70]$ *(the weight vector) and* $u = [82\ 26\ 42\ 36\ 70\ 10\ 52\ 17]$ *(the utility vector). The solutions, in the term of the knapsack capacity,* C*, are reported in Table 2.*

Tabela 2: Example 5.2.

C	Optimal Solution	Optimal Value
52	(1,0,0,0,1,0,0,0)	155
65	(1,0,0,0,1,0,1,0)	204
100	(1,0,0,1,1,1,1,0)	250
132	(1,0,1,1,1,1,1,0)	292
145	(1,1,1,1,1,0,1,0)	308
220	(1,1,1,1,1,0,1,1)	325
228	(1,1,1,1,1,0,1,1)	325

6. Conclusions

In this work we proposed a continuous optimization approach for solving the 0–1 knapsack problem. In the continuous space, the problem was reformulated as a convex generalized multiplicative programming problem. By using convex analysis results, the problem was reformulated in the outcome space as an optimization problem with infinitely many linear inequality constraints, and then solved through a relaxation branch–and–bound algorithm. Experimental results have attested the viability and efficiency of the proposed global optimization algorithm, which is, in addition, easily programmed through standard optimization packages.

The extension of the proposed algorithm for solving the related integer optimization problems, including other classic problems of the family of knapsack problems, is under current investigation by the authors.

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